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A comment on the ideal relativistic Bose gas

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Abstract. Rigorous results are proved which are directly applicable to the problem of writing the thermodynamic properties of an ideal relativistic Bose gas in terms of modified Bessel functions of the second kind. The results are particularly important when the Bose gas is at, or below, its condensation temperature.

1. Introduction

With the recent discussion of the cosmological implications of a massive primordial photon gas by Kuzmin and Shaposhnikov (1979), and with the attention being paid to problems associated with quarks and quark confinement (for a review of statistical mechanics at high energy density see Sertorio (1979)), has come a revival of interest in the ideal relativistic Bose gas (Beckmann *et al* 1979, Aragão de Carvalho and Goulart Rosa 1980).

In the paper by Beckmann *et al*, which pays particular attention to the phenomenon of Bose condensation, evaluation of the logarithm of the grand partition function is achieved through recourse to a Taylor series expansion of $\lg\{1-\exp[\beta(\mu-e-e_0)]\}$ (where e_0 is the rest energy), with subsequent use of the integral representation for the modified Bessel functions of the second kind. This technique is quite acceptable when the chemical potential, μ, does not equal e_0. However, for a Bose gas at or below its condensation temperature, $\mu=e_0$ and, under these circumstances, the technique needs further examination since it involves expanding $\lg[1-\exp(-\beta e)]$ over a range including the value $e=0$.

In the paper by Aragão de Carvalho and Goulart Rosa, use is made of a technique involving Mellin transforms to derive the basic equations. Again the resulting expressions are written in terms of modified Bessel functions of the second kind. However, once again, the method, though perfectly acceptable for a Bose gas above its condensation temperature, would involve the expansion of $\lg[1-\exp(-\beta e)]$ over a range including $e=0$ if a Bose gas at or below its condensation temperature is considered.

It is the purpose of the present article to show that the above procedures are valid for use with a Bose gas at and below its condensation temperature. To this end, two general mathematical results will be proved in § 2 and their application to the methods of Beckman *et al* and Aragão de Carvalho and Goulart Rosa will be indicated briefly in § 3.

2. Two lemmas on integration of an infinite series

Each of the following results may be derived by a straightforward application of the Lebesgue dominated convergence theorem. However, it may be useful to present proofs which are self-contained and independent of the apparatus of measure theory.

Lemma 1. Let $\{g_n\}$ be a sequence of non-negative functions such that the series $\Sigma_{n=1}^{\infty} g_n(x)$ converges uniformly, in any interval (ε, ∞) with $\varepsilon > 1$, to a limit $G(x)$. Let F be a non-negative function satisfying $\int_\varepsilon^\infty F(x)\,\mathrm{d}x < \infty$ for any $\varepsilon > 1$. Then, for $\varepsilon > 1$,

$$\int_\varepsilon^\infty F(x)G(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_\varepsilon^\infty F(x)g_n(x)\,\mathrm{d}x.$$

Proof.

$$\sum_{n=1}^{m}\int_\varepsilon^\infty F(x)g_n(x)\,\mathrm{d}x = \int_\varepsilon^\infty \left(F(x)\sum_{n=1}^{m} g_n(x)\right)\mathrm{d}x$$

$$= \int_\varepsilon^\infty F(x)G(x)\,\mathrm{d}x + \int_\varepsilon^\infty F(x)\left(\sum_{n=1}^{m} g_n(x) - G(x)\right)\mathrm{d}x$$

where the second integral on the right-hand side is bounded in absolute value by

$$\int_\varepsilon^\infty F(x)\left|\sum_{n=1}^{m} g_n(x) - G(x)\right|\mathrm{d}x.$$

Since convergence is uniform, given $\delta > 0$, $m(\delta)$ may be chosen sufficiently large so that

$$\left|\sum_{n=1}^{m} g_n(x) - G(x)\right| < \delta \qquad \text{for } \varepsilon \leqslant x < \infty.$$

In that case, the integral is bounded by $\delta\int_\varepsilon^\infty F(x)\,\mathrm{d}x$ and hence converges to zero as $m \to \infty$.

The required result follows.

Lemma 2. Under the same assumptions as for lemma 1,

$$\lim_{\varepsilon\to 1}\int_\varepsilon^\infty F(x)G(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_1^\infty F(x)g_n(x)\,\mathrm{d}x$$

(that is, the equation holds whenever either side is finite).

Proof.

(i) $$\lim_{\varepsilon\to 1}\int_\varepsilon^\infty F(x)G(x)\,\mathrm{d}x = \lim_{\varepsilon\to 1}\sum_{n=1}^{\infty}\int_\varepsilon^\infty F(x)g_n(x)\,\mathrm{d}x \qquad \text{(by lemma 1)}$$

$$\geqslant \lim_{\varepsilon\to 1}\sum_{n=1}^{N}\int_\varepsilon^\infty F(x)g_n(x)\,\mathrm{d}x$$

$$= \sum_{n=1}^{N}\int_1^\infty F(x)g_n(x)\,\mathrm{d}x.$$

This inequality holds for all N, whereas the original limit is independent of N, and so, let $N\to\infty$ to obtain

$$\lim_{\varepsilon\to 1}\int_\varepsilon^\infty F(x)G(x)\,\mathrm{d}x \geqslant \sum_{n=1}^{\infty}\int_1^\infty F(x)g_n(x)\,\mathrm{d}x. \tag{2.1}$$

(ii) Alternatively,

$$\lim_{\varepsilon\to 1}\int_\varepsilon^\infty F(x)G(x)\,\mathrm{d}x = \lim_{\varepsilon\to 1}\sum_{n=1}^{\infty}\int_\varepsilon^\infty F(x)g_n(x)\,\mathrm{d}x$$

$$\leqslant \sum_{n=1}^{\infty}\int_1^\infty F(x)g_n(x)\,\mathrm{d}x. \tag{2.2}$$

The required result follows on comparing (2.1) and (2.2).

3. Application of results

In Beckmann *et al* (1979), a system of relativistic bosons, each with a rest energy e_0 and the relativistic energy spectrum

$$e=(e_0^2+c^2p^2)^{1/2},$$

contained in an n-dimensional spatial volume V, is considered. The logarithm of the grand partition function is given by

$$\lg\Xi=\lg\{1-\exp[\beta(\mu-e_0)]\}+\frac{2\pi^{n/2}V\beta}{nh^nc^n\Gamma(n/2)}\int_{e_0}^\infty\frac{(e^2-e_0^2)^{n/2}\,\mathrm{d}e}{\exp[(e-\mu)\beta]-1}$$

where μ is the chemical potential and $\beta\equiv 1/kT$.

In the integral, effect the change of variable $t=e/e_0$ to give

$$\int_1^\infty\frac{(t^2-1)^{n/2}\,\mathrm{d}t}{\exp[\beta(e_0t-\mu)]-1}.$$

At and below the condensation temperature, $\mu=e_0$, and so for this region consider

$$\lim_{\varepsilon\to 1}\int_\varepsilon^\infty\frac{(t^2-1)^{n/2}\,\mathrm{d}t}{\exp[\beta e_0(t-1)]-1}.$$

By using the results of the previous section, together with the integral representation of the modified Bessel functions of the second kind (Watson 1966), it is seen that this latter integral may be written

$$\sum_{k=1}^{\infty}\exp(k\beta e_0)\frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})(\frac{1}{2}k\beta e_0)^{(n+1)/2}}K_{(n+1)/2}(k\beta e_0).$$

This expression is seen to agree with that derived by Beckmann *et al* when the Bose gas is at or below its condensation temperature.

In Aragão de Carvalho and Goulart Rosa (1980), the results given in the previous section of this present article are needed to justify the equation they use as the starting point for their discussion, namely the writing of the logarithm of the grand partition function of the ideal boson gas as an integral transform of the single-particle partition function. It should be pointed out that they do not discuss Bose condensation, but to make their formulation applicable when the gas is at or below its condensation temperature, it is necessary to show that the Mellin transform for $\lg(1-x)$, $0<x<1$, (Erdelyi 1954) holds when $x=\exp(-\beta e)$ even when $e=0$. This follows by applying the earlier results.

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